Quiz 4 Solution

Tuesday, September 17, 2013 3:44 PM

① Suppose $P_{X,Y} = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}$. Find $I(X;Y)$. ≈ 0.02

(4.5 pt)

$$H(X,Y) = H\left(\left[\tfrac{1}{5}\ \tfrac{2}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\right]\right) = 1.9219$$

$$\begin{array}{ccc} & {}^{x}\backslash{}^{y} & \\ P_{X,Y} = & \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix} & \begin{array}{l} \to 3/5 \\ \to 2/5 \end{array} \Big\} \Rightarrow H(X) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) = 0.9710 \\ & \begin{array}{cc} \downarrow & \downarrow \\ \tfrac{2}{5} & \tfrac{3}{5} \end{array} & \Rightarrow H(Y) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) = 0.9710 \end{array}$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.019973 \approx 0.0200.$$

② Suppose $p_X(x) = \begin{cases} 1/5, & x = 0 \\ 4/5, & x = 1 \\ 0, & \text{otherwise} \end{cases}$ and

$P_{Y|X}$

| x \ y | 0 | 1 |
|---|---|---|
| 0 | 1/5 | 4/5 |
| 1 | 2/5 | 3/5 |

(4.5 pt)

Find $I(X;Y)$. ≈ 0.0215

$P_{Y|X}$

| x \ y | 0 | 1 |
|---|---|---|
| 0 | 1/5 | 4/5 |
| 1 | 2/5 | 3/5 |

$\to H(Y|0) = H\left(\left[\tfrac{1}{5}\ \tfrac{4}{5}\right]\right) \approx 0.7219$

$\to H(Y|1) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) \approx 0.9710$

$\Rightarrow H(Y|X) = \tfrac{1}{5} \times H(Y|0) + \tfrac{4}{5} \times H(Y|1) \approx 0.9211$

$$\underbrace{\begin{bmatrix} 1/5 & 4/5 \end{bmatrix}}_{P_X} \underbrace{\begin{bmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{bmatrix}}_{P_{Y|X}} = \underbrace{\begin{bmatrix} \tfrac{9}{25} & \tfrac{16}{25} \end{bmatrix}}_{P_Y} \Rightarrow H(Y) = H\left(\left[\tfrac{9}{25}\ \tfrac{16}{25}\right]\right) \approx 0.9427$$

$$I(X;Y) = H(Y) - H(Y|X) \approx 0.0215$$

Alternatively,

$P_{Y|X}$

| x \ y | 0 | 1 |
|---|---|---|
| 0 | 1/5 | 4/5 |
| 1 | 2/5 | 3/5 |

$\xrightarrow{\times \frac{1}{5}}$ (row 0), $\xrightarrow{\times \frac{4}{5}}$ (row 1)

| x \ y | 0 | 1 |
|---|---|---|
| 0 | 1/25 | 4/25 |
| 1 | 8/25 | 12/25 |

$$H(X,Y) = H\left(\left[\tfrac{1}{25}\ \tfrac{4}{25}\ \tfrac{8}{25}\ \tfrac{12}{25}\right]\right) \approx 1.6431$$

$$\begin{array}{c|cc} 0 & 1/5 & 4/5 \\ 1 & 2/5 & 3/5 \end{array} \quad \xrightarrow[\times \frac{4}{5}]{} \quad \begin{array}{c|cc} 0 & \overline{25} & \overline{25} \\ 1 & \frac{8}{25} & \frac{12}{25} \end{array}$$

$H(X,Y) = H([\,\ldots\,])$

$\approx 1.6431$

$$\begin{array}{cc} \downarrow & \downarrow \\ \frac{9}{25} & \frac{16}{25} \end{array} \Rightarrow H(Y) = H\left(\left[\tfrac{9}{25} \;\; \tfrac{16}{25}\right]\right) \approx 0.9427$$

$H(X) = H\left(\left[\tfrac{1}{5} \;\; \tfrac{4}{5}\right]\right)$

$\approx 0.7219$

$I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.0215$

*③ Suppose $p_{Y|X}$ : $\begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 4/5 \end{bmatrix}$ Find $I(X;Y)$.

(1 pt)

Remark: Normally, to calculate $I(X;Y)$ you will need both $p_X$ and $p_{Y|X}$.
Here, there must be something special about $p_{Y|X}$ that allows you to get $I(X;Y)$ without $p_X$.

Intuition: The rows of $p_{Y|X}$ are all the same. This implies that knowing the value of $x$ does not change the pmf of $Y$.

Since $X$ does not give any information about $Y$, we expect $I(X;Y) = 0$.

Direct calculation:

$H(Y|x) = H\left(\left[\tfrac{1}{5} \;\; \tfrac{4}{5}\right]\right) \approx 0.7219$ for any $x$.

So, $H(Y|X) = \sum_x p_X(x) H(Y|x) \approx 0.7219 \underbrace{\sum_x p_X(x)}_{1} \approx 0.7219$.

$I(X;Y) = H(Y) - H(Y|X)$ So, we need $H(Y)$ which in turn need $p_Y$.

Let's try $p_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\underset{p_X}{[1-p \;\;\; p]} \underset{p_{Y|X}}{\begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 4/5 \end{bmatrix}} = \underset{p_Y}{\left[\tfrac{1}{5} \;\;\; \tfrac{4}{5}\right]} \Rightarrow H(Y) = H\left(\left[\tfrac{1}{5} \;\; \tfrac{4}{5}\right]\right) = H(Y|X)$$

regardless of the value of $p$

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.